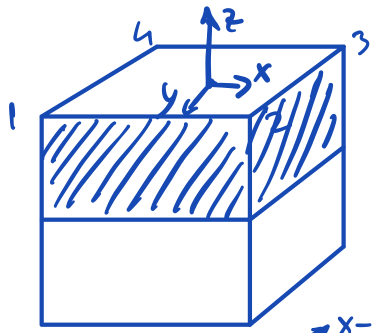
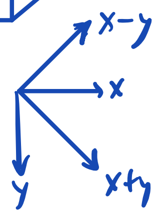
Answer model exam Symmetry in Physics of April 1, '25

1a) coloured block

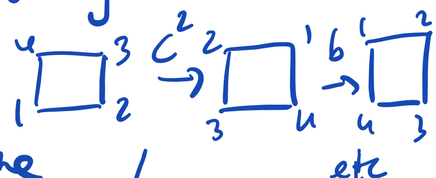




e

c = rotation around $\hat{z}$ over 90°

c^2, c^3



b = reflection in zy plane

b_2 = reflection in zx plane $= bc^2$

b_3 = " $(x-y)-z$ plane $= bc$

b_4 = " in $(x+y)-z$ plane $= bc^3$

$$G_B = gp\{b,c\} \text{, with } b^2 = e = (bc)^2 = c^4 = e$$

$G_B < O(3)$ hence C_{4v} ($\cong D_4$)

1b) conjugacy classes: $(e)=\{e\}$ $\quad (c)=\{c,c^3\}$ (conjugate via b) $\quad (c^2)=\{c^2\}$

$(b)=\{\underbrace{b, bc^2}_{\text{conj via } c}\}$ $\quad (bc)=\{\underbrace{bc, bc^3}_{\text{conj. via } c}\}$

$b \not\sim bc$ no 45° rotation in G_B

$c^2 \not\sim b$, etc 180° rotation $\neq$ reflections

5 classes $\Rightarrow$ 5 irreps

$$\sum_{\mu=1}^{5} n_\mu^2 = 8 \qquad n_1=n_2=n_3=n_4=1, \quad n_5=2$$

	(e)	(c)	(c^2)	(b)	(bc)
$D^{(1)}$	1	1	1	1	1
$D^{(2)}$	1	1	1	−1	−1
$D^{(3)}$	1	−1	1	1	−1
$D^{(4)}$	1	−1	1	−1	1
$D^{(5)}$	2	a = 0	b = −2	c = 0	d = 0

$\Leftarrow$ due to orthogonality, e.g.

$1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 2b = 0 \Rightarrow b = -2$

1D irrep: $\chi(c)^4 = 1$

$\Rightarrow \chi(c) \in \{\pm 1, \pm i\}$

$\chi(b)^2 = 1 \Rightarrow \chi(b) = \pm 1$

$\chi(bc)^2 = 1 \Rightarrow \chi(c)^2 = 1$

$\Rightarrow \chi(c) = \pm 1$

1c) D^V: $\quad \chi^V(e) = 3 \quad \chi^V(c) = 1 + 2\cos 90^\circ = 1$

$\chi^V(c^2) = 1 + 2\cos 180^\circ = -1$

$$D^V(b) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \chi^V(b) = 1$$

$$D^V(bc) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$\chi^V(bc) = 1$$

$$\chi^V = (3, 1, -1, 1, 1) \Rightarrow \chi^V = \chi^{(1)} + \chi^{(5)}$$

or $\langle \chi^V, \chi^{(1)} \rangle = \frac{1}{8}\left(3 + 2\cdot 1 - 1\cdot 1 + 2\cdot 1 + 2\cdot 1\right) = 1$

$\Rightarrow$ EDM possible/allowed $\Rightarrow$ MDM not

(check $\chi^A = (3, 1, -1, -1, -1)$: $\langle \chi^A, \chi^{(1)} \rangle = \frac{1}{8}(3+2-1-4)$
$= 0$ ✓)

2a) $\sigma'_{ij} = D^V_{ik} D^V_{jl} \sigma_{kl}$ in general

If $\sigma' = \sigma \Rightarrow \sigma_{ij} = D^V_{ik} \sigma_{kl} (D^{VT})_{lj}$

$\Rightarrow \sigma = D^V \sigma D^{VT}$ (matrix notation)

$\Rightarrow \sigma D^V = D^V \sigma \underbrace{D^{VT} D^V}_{\mathbb{1}} = D^V \sigma$

$\mathbb{1}$ since $D^{VT} = D^{V-1}$

$\Rightarrow [D^V, \sigma] = 0$

2b) $\sigma = \begin{pmatrix} a & & 0 \\ & a & \\ 0 & & a+b \end{pmatrix}$ invariant for subgp of $O(3)$

Consider a general 3×3 (real) matrix : $\begin{pmatrix} c & d & e \\ f & g & h \\ k & l & m \end{pmatrix} \equiv O$

demand that it commutes with σ:

$$O\sigma = \begin{pmatrix} ca & da & e(a+b) \\ fa & ga & h(a+b) \\ ka & la & m(a+b) \end{pmatrix} = \sigma O = \begin{pmatrix} ac & ad & ae \\ af & ag & ah \\ (a+b)k & (a+b)l & (a+b)m \end{pmatrix}$$

$\Rightarrow e = 0 = h = k = l \quad \Rightarrow \begin{pmatrix} c & d & 0 \\ f & g & 0 \\ 0 & 0 & m \end{pmatrix}$

demand that it is in $O(3)$: $\begin{pmatrix} c & d \\ f & g \end{pmatrix} \in O(2)$ & $m = \pm 1$

2c) $C_{4v} < O(3)$ $\quad D^v(c) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ & $D^v(b) = \begin{pmatrix} -1 & & \Theta \\ & 1 & \\ \Theta & & 1 \end{pmatrix}$

$\sigma = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad [D^v(c), \sigma] = 0 \Rightarrow \begin{pmatrix} b & -a & c \\ e & -d & f \\ h & -g & i \end{pmatrix} = \begin{pmatrix} -d & -e & -f \\ a & b & c \\ g & h & i \end{pmatrix}$

$\Rightarrow b = -d, \; a = e, \; c = f = 0 = g = h$

$\Rightarrow \sigma = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & i \end{pmatrix} \quad [D^v(b), \sigma] = 0$

$\Rightarrow \begin{pmatrix} -a & b & 0 \\ b & a & 0 \\ 0 & 0 & i \end{pmatrix} = \begin{pmatrix} -a & -b & 0 \\ -b & a & 0 \\ 0 & 0 & i \end{pmatrix}$

$\Rightarrow b = 0$: $\quad \sigma = \begin{pmatrix} a & & \Theta \\ & a & \\ \Theta & & i \end{pmatrix}$ indeed like in (b) for $i \equiv (a+b)$

20% $\left\{ \; D^v \text{ is of the form } \begin{pmatrix} O(2) & & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & \pm 1 \end{pmatrix} \right.$, C_{4v} is subgp that according to (b) should leave $\begin{pmatrix} a & & \\ & a & \\ & & a+b \end{pmatrix}$ invariant and it does

3a) $\langle \tfrac{3}{2} m_s' | S_z | \tfrac{3}{2} m_s \rangle$ in matrix form:

$$\hbar \begin{pmatrix} \frac{3}{2} & & & \Theta \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ \Theta & & & -\frac{3}{2} \end{pmatrix}$$

3b) $\langle \frac{3}{2}\, m_s' \,|\, \exp\left(\frac{i}{\hbar}\theta S_z\right) \,|\, \frac{3}{2}\, m_s \rangle$ in matrix form:

$$\begin{pmatrix} e^{i\frac{3}{2}\theta} & & & \\ & e^{i\frac{\theta}{2}} & & \\ & & e^{-i\frac{\theta}{2}} & \\ & & & e^{-i\frac{3\theta}{2}} \end{pmatrix} \qquad \theta \in [0, 4\pi]$$

end points identified

3c) $V^{(s)} = \{ |s,s\rangle, \ldots, |s,-s\rangle \}$ is a $2s+1$-D space that is invariant under SU(2) transformations

50%

$$\left[\vec{S}^2, \exp\left(\frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{S}\right)\right] = 0 \quad \forall \theta, \hat{n}$$

$\Rightarrow$ s quantum number cannot be changed.

$\Rightarrow$ $V^{(s)} \to V^{(s)}$ under SU(2) transformations

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Also, $V^{(s)}$ has no smaller invariant subspaces

since $|s, m_s\rangle \to |s, m_s'\rangle$ $\forall m_s, m_s'$

by some combination of $U(\theta, \hat{x})$ & $U(\theta', \hat{y})$

$\Rightarrow$ $V^{(s)}$ carrier space of an irrep of SU(2)

All 9 subquestions are of equal weight: 9 points in total

Grade is sum of points + 1, rounded off

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Groningen, 26-3-'25